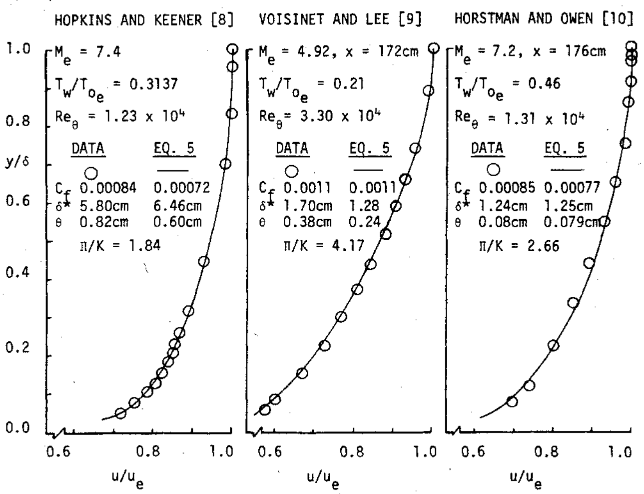


**Fig.1 Velocity distributions in boundary layer with heat transfer.**

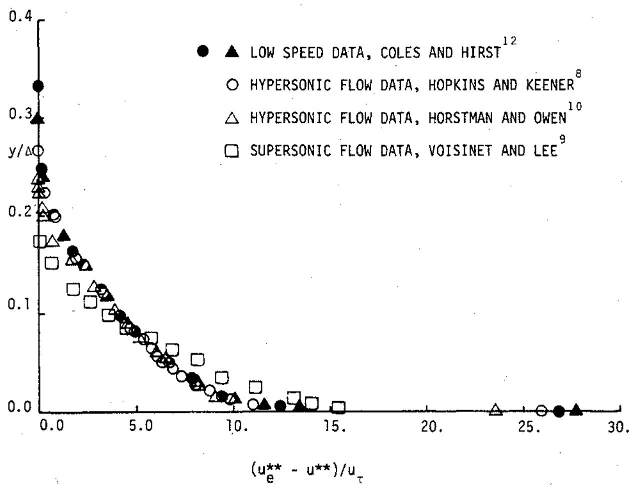


**Fig.2 Comparison of generalized velocity defect for low-speed flow and compressible flow with heat transfer.**

different. The values varied considerably from profile to profile for all of the data examined and generally tended to be higher than for low-speed or compressible adiabatic flows.

The experimental profiles also may be plotted in the form $(u_e^{**}-u^{**})/u_\tau$ vs $y/\Delta$, where $\Delta$ is analogous to the defect thickness used by Gran et al.,[4] in their comparison of cold-wall, high-speed data with low-speed data[12]. Such a plot is shown in Fig.2. Although the profile from Ref.9 deviates from the low-speed results, the profiles from Refs. 8 and 10 correspond quite closely with the low-speed data. This suggests that the latter two profiles were nearer equilibrium than the former, which, in turn, may account for the fact that the value of $\delta$ determined for the profile from Ref. 9 does not correspond as closely to the viscid-inviscid flow boundary as was found for the other two profiles.

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# General Thermal Constriction Parameter for Annular Contacts on Circular Flux Tubes

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## Introduction

In a recent paper,[1] a general expression was obtained for constriction resistances due to arbitrary flux distributions over circular contact areas on a circular flux tube. By means of the general expression, special cases such as the isothermal and constant flux boundary condition could be evaluated. This paper extends that mathematical development to the more general case of an annular contact area supplying heat to a coaxial circular flux tube.

## Problem Statement and Solution

An annular contact area of radii $a$, $b$ $(a < b)$ is situated on the end of a solid circular cylinder of radius $c$ (Fig. 1). The long cylinder, whose thermal conductivity is $k$, is perfectly insulated except for the annular contact area where the flux is prescribed. For the analysis to follow, a circular cylinder coordinate system $(r, z)$ is chosen, and the origin is placed on the axis of the cylinder.

Since there is steady heat flow through the cylinder, the governing equation is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{1}$$

and the boundary conditions are

$$z=0, \qquad 0 \le r < a, \qquad \partial T/\partial z = 0 \tag{2a}$$

$$a < r < b, \qquad -k(\partial T/\partial z) = f(r) \tag{2b}$$

$$b < r \le c, \qquad \partial T/\partial z = 0 \tag{2c}$$

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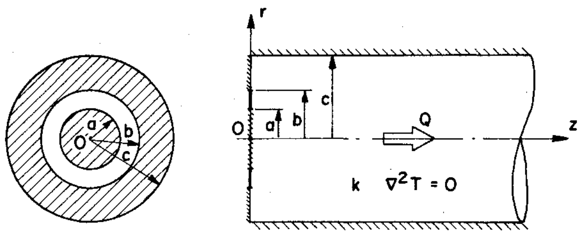


**Fig. 1 Annular contact on a circular heat flux tube.**

$$z \to \infty, \quad 0 \le r \le c, \quad -k(\partial T/\partial z) = Q/\pi c^2 \tag{3}$$
$$r=0, \quad z \ge 0, \quad \partial T/\partial r = 0 \text{ by symmetry} \tag{4}$$
$$r=c, \quad z \ge 0, \quad \partial T/\partial r = 0 \tag{5}$$

In Eq. (3), $Q$ is the total heat flow rate through the cylinder. Following the method of solution in Ref. 1, we immediately write the temperature distribution as

$$T = \frac{-Qz}{k\pi c^2} + C_0 + \sum_{n=1}^{\infty} C_n e^{-\lambda_n z} J_0(\lambda_n r) \tag{6}$$

It can be shown that Eq. (6) satisfies the differential equation, as well as the boundary conditions given by Eqs. (3) and (4). If we choose $(\lambda_n c)$ to be the roots of $J_1(\lambda_n c) = 0$, then Eq. (5) will be satisfied as well. The coefficients $C_0$ and $C_n$ will be obtained next.

The average temperature in any section $z$ is defined as

$$\bar{T}(z) = \frac{1}{\pi c^2} \int_0^c T\, 2\pi r dr \tag{7}$$

After substitution of Eq. (6) into Eq. (7), one obtains[1]

$$\bar{T}(z) = \frac{-Qz}{k\pi c^2} + C_0 \tag{8}$$

It is evident that, when $z=0$, $\bar{T}(0) = C_0$. Thus, $C_0$ is the average temperature of the contact plane. The difference between the average temperature of the contact plane and the average temperature of some arbitrary plane $z = \ell$, where $\ell >> c$, is simply

$$\bar{T}(0) - \bar{T}(\ell) = Q\ell / k\pi c^2 \tag{9}$$

The temperature drop given in Eq. (9) is due to uniform heat flow through a right circular cylinder of length $\ell$, flow area $\pi c^2$, and thermal conductivity $k$. If this temperature drop is divided by $Q$, we have the cylinder resistance $R_0 = \ell / k\pi c^2$.

The average temperature of the contact area is defined as

$$\bar{T}_c = \frac{1}{\pi(b^2 - a^2)} \int_a^b T\, 2\pi r dr \tag{10}$$

After substitution of Eq. (6) into Eq. (10), one obtains

$$\bar{T}_c = C_0 + \frac{2}{(b^2-a^2)} \sum_{n=1}^{\infty} C_n \int_a^b r\, J_0(\lambda_n r) dr \tag{11}$$

The integral in Eq. (11) can be evaluated using the properties of Bessel functions,[2] and Eq. (11) becomes

$$\bar{T}_c = C_0 + \frac{2}{(b^2-a^2)} \sum_{n=1}^{\infty} \frac{C_n}{\lambda_n^2} \{ (\lambda_n b) J_1(\lambda_n b) - (\lambda_n a) J_1(\lambda_n a) \} \tag{12}$$

We see from Eq. (12) that the average temperature of the annular contact is equal to the average temperature of the contact plane plus a difference given by the infinite series. This difference, as will be shown, is a manifestation of the constriction resistance.

The temperature drop from the average contact area temperature to the average temperature in some arbitrary plane $z = \ell$ is assumed to be due to the constriction resistance and the cylinder resistance. Thus,

$$\bar{T}_c - \bar{T}(z=\ell) = QR_t = Q(R_0 + R_a) \tag{13}$$

where $R_a$ is defined as the constriction resistance due to the annular contact area.

By means of Eqs. (8) and (12), Eq. (13) yields

$$R_a = \frac{2}{Q(b^2-a^2)} \sum_{n=1}^{\infty} \frac{C_n}{\lambda_n^2} \{ (\lambda_n b) J_1(\lambda_n b) - (\lambda_n a) J_1(\lambda_n a) \} \tag{14}$$

for the constriction resistance, which is dependent upon the coefficients $C_n$. These coefficients can be determined by considering the boundary conditions given by Eq. (2).

According to Eq. (6), along $z=0$ we have

$$\frac{\partial T}{\partial z} = \frac{Q}{k\pi c^2} + \sum_{n=1}^{\infty} C_n \lambda_n J_0(\lambda_n r) \tag{15}$$

Multiplying Eq. (15) through by $r\, J_0(\lambda_m r)\, dr$ and integrating with respect to $r$ from 0 to $c$, we obtain

$$\int_0^c -\frac{\partial T}{\partial z} r J_0(\lambda_m r) dr = \frac{Q}{k\pi c^2} \int_0^c r J_0(\lambda_m r) dr + \sum_{n=1}^{\infty} C_n \lambda_n \int_0^c r J_0(\lambda_n r) J_0(\lambda_m r) dr \tag{16}$$

Utilizing the orthogonal properties of Bessel functions, it can be shown that the first integral on the right-hand side of Eq. (16) is zero at both limits because $J_1(0) = 0$ and $J_1(\lambda_n c) = 0$ by Eq. (5). The second integral is zero when $m \ne n$; otherwise we have

$$\tfrac{1}{2} (C_n / \lambda_n) (\lambda_n c)^2 J_0^2(\lambda_n c) \tag{17}$$

when $m = n$.

Since the integral on the left-hand side is zero over the ranges $0 \le r < a$ and $b < r \le c$, we have for the coefficients the following relationship:

$$C_n = \frac{2\lambda_n}{(\lambda_n c)^2 J_0^2(\lambda_n c)} \int_a^b -\frac{\partial T}{\partial z} r J_0(\lambda_n r) dr \tag{18}$$

The coefficients are clearly a function of the temperature gradient (flux distribution) over the annular contact area. Upon substitution of Eq. (18) into Eq. (14), we have the relation between the constriction resistance and the temperature gradient over the contact area.

If we write

$$-(\partial T / \partial z) = \kappa f(r) \tag{19}$$

where $\kappa$ is some constant, it can be shown that

$$\kappa = Q / 2\pi k \int_0^b r f(r) dr \tag{20}$$

Before proceeding with the determination of the expression for the dimensionless constriction resistance, it is advantageous to define some dimensionless geometric ratios:

$$\epsilon = a/b, \quad 0 \le \epsilon < 1 \tag{21a}$$

$$\alpha = b/c, \quad 0 < \alpha < 1 \tag{21b}$$

Thus it can be shown that

$$\lambda_n b = \lambda_n c\ \alpha = \alpha\delta_n \quad (22a)$$

$$\lambda_n a = \lambda_n c\ \epsilon\alpha = \alpha\epsilon\delta_n \quad (22b)$$

where $\delta_n = \lambda_n c$, the roots of $J_1(\delta_n) = 0$.

If we further define $R_a^{*} = kbR_a$ and $u = r/b$, then we can show that

$$R_a^{*} = \frac{(2/\pi)}{(1-\epsilon^2)\int_\epsilon^1 uf(u)du} \sum_{n=1}^{\infty} \frac{J_1(\alpha\delta_n)\left\{1-\epsilon\frac{J_1(\alpha\epsilon\delta_n)}{J_1(\alpha\delta_n)}\right\}}{\delta_n^2 J_0^2(\delta_n)} \int_\epsilon^1 uf(u)J_0(\alpha\delta_n u)du \quad (23)$$

is the general expression for the thermal constriction resistance due to an annular contact area with an arbitrary flux distribution over the contact area. $R_a^{*}$ often is called the constriction parameter and is defined as $\psi_a$.

Equation (23) is valid for any axially symmetric flux distribution and can be evaluated analytically or numerically. For the case of a uniform heat flux, $f(u) = 1$, we obtain

$$\int_\epsilon^1 uf(u)\, du = \tfrac{1}{2}(1-\epsilon^2) \quad (24)$$

$$\int_\epsilon^1 uf(u)J_0(\alpha\delta_n u)du = \frac{1}{(\alpha\delta_n)^2}\{(\alpha\delta_n)J_1(\alpha\delta_n) - (\alpha\epsilon\delta_n)J_1(\alpha\epsilon\delta_n)\} \quad (25)$$

With Eqs. (24) and (25), Eq. (23) becomes

$$\psi_a = \frac{(4/\pi)}{(1-\epsilon^2)^2} \sum_{n=1}^{\infty} \frac{J_1^2(\alpha\delta_n)\left\{1-\epsilon\frac{J_1(\alpha\epsilon\delta_n)}{J_1(\alpha\delta_n)}\right\}^2}{\delta_n^3 J_0^2(\delta_n)} \quad (26)$$

in agreement with the analysis of Yip.[4] In his dissertation, Yip presents a plot of $\psi_a$ for various values of $\epsilon$ and $\alpha$. It can be seen that the general expression developed here reduces to the general expression valid for a circular contact area[1] when $\epsilon$ is set equal to zero.

## Conclusions

A general annular constriction parameter valid for axially symmetric flux distributions has been derived from first principles. In the limiting case of a circular contact, the general expression developed here reduces to one developed previously. The annular constriction parameter for the case of uniform heat flux is seen to be a particular case of the general case, and it was obtained with great ease from the general expression.

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# Notes on Finite Elements for Nearly Incompressible Materials

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## Introduction

IN applying conventional finite element methods to incompressible materials, the resulting element stiffness matrix will be infinite. Alternative formulations must be provided whereby the mean stresses in the elements are introduced as unknowns in addition to the nodal displacements. Herrmann[1] and Key[2] introduced such remedies for the assumed displacement method, while Tong[3] and Lee[4] provided corresponding schemes for the assumed stress hybrid model. For nearly incompressible materials it is also known that the conventional finite element models based on the assumed displacement approach do not give reliable solutions, unless the mean stresses are retained as additional variables. Nagtegaal et al.,[5] in explaining the failure of conventional finite element solutions in the fully plastic range by the tangent stiffness approach, have pointed out the severe kinematic constraints on the modes of deformation introduced by the zero dilatation condition. They have demonstrated that a plane strain element made of four constant strain triangles forming two diagonals can give satisfactory solutions for the fully plastic range.

The purpose of this Note is to point out that the incompressibility constraints imposed on the assumed stress hybrid elements are much less severe. Hence, solutions for materials very close to incompressibility, can be obtained with sufficient accuracy by such elements using the conventional method which contains only nodal displacements as unknowns. This Note will also point out the situations for which the formulations by Key can also be converted to the conventional matrix displacement methods for nearly incompressible materials. In the following developments, only isotropic materials will be considered and the effect of body forces will be excluded for the sake of simplicity.

## Incompressibility

The pointwise incompressibility condition i.e., the condition of vanishing dilatational strain, provides certain constraints to the assumed displacement finite element formulation. As an example the displacements for a four node rectangular plane strain element can be assumed as

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy \quad (1a)$$

$$v = \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy \quad (1b)$$

However, to satisfy the condition $\epsilon_x + \epsilon_y = (\partial u/\partial x) + (\partial v/\partial y) = 0$, it is necessary that

$$\alpha_2 + \alpha_7 = 0 \quad (2a)$$

$$\alpha_4 = 0 \quad (2b)$$

$$\alpha_8 = 0 \quad (2c)$$

These three conditions of constraint will limit the element motion to only two possible deformation modes which, of course, cannot represent more general state of strain.

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